Dynamical properties of hard-core anyons in one-dimensional optical lattices

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We investigate the dynamical properties of anyons confined in one-dimensional optical lattice combined with a weak harmonic trap using the exact numerical method based on a generalized Jordan-Wigner transformation. The density profiles, momentum distribution, occupation distribution and occupations of the lowest natural orbital are obtained for different statistical parameters. The density profiles of anyons display the same behaviors irrespective of statistical parameter in the full evolving period. While the behaviors dependent on statistical property are shown in the momentum distributions and occupations of natural orbitals.

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I. INTRODUCTION

Generally according to the quantum statistical property particle is classified as boson and fermion. The wavefunction of identical bosons is symmetric under exchange while that of identical fermions is antisymmetric under exchange. As a natural generalization, physicist proposed that there exists anyon interpolating between Bose and Fermi statistics, which satisfies fractional statistics [1]. It has become an important concept in condensed matter physics [2–4] and has ever been used for successfully explaining the fractional quantum Hall effect (FQHE) [5]. Fractional statistics also play important roles in the theory for non-fermi liquid, Chern-Simon theory, and other aspects [6, 7]. Another potential application of anyons is to realize the topological quantum computation with non-abelian anyons [8, 9]. Besides the traditional two dimensional electron system, cold atoms in low dimension is also a popular platform to realize the fractional statistics. It has been suggested that anyons can be created, detected and manipulated in rotating Bose-Einstein condensates (BECs) and cold atoms in optical lattices [10–12]. Particularly, researches on anyons in cold atoms are not restricted in two dimensional system but the proposal to realize anyons in one dimensional (1D) optical lattice is also put forward recently [13].

Since BECs are realized experimentally, cold atoms has been making great progress both experimentally and theoretically for its “purity” and high controllability comparing with the traditional condensed matter system. For the dimensionality effect and extremely profound strong correlation in low dimensional system, the quantum gas in low dimension has also attract much attention [14]. With anisotropic magnetic trap or two-dimensional optical lattice potentials, the particle motion is tightly confined in two directions to zero point oscillations [15–17] and the strongly correlated Tonks-Girardeau (TG) regime can be achieved [16, 17]. By crossing the confinement-induced resonance (CIR) from the TG gas, the super TG (sTG) gas is accessible, which show stronger correlation than TG gas [18]. The interaction between atoms can be tuned in the full interacting regime with Feshbach resonance technique and confinement-induced resonance by tuning magnetic field. It is the excellent tunability of cold atom that makes it to be an excellent candidate to realize fractional statistics.

In fact, before the experimental scheme of Keilmann et. al. [13], 1D anyon gas has been investigated theoretically in various 1D systems [19–22] including the Bose quantum gas. Kundu proved that a 1D Bose gas interacting through δ-function potential combined with double δ-function potential and derivative δ-function potential is equivalent to the anyon gas interacting via δ-function potential [21]. This stimulated many research interests on δ-anyon gas [22–32]. It turns out that the ground state density distribution of δ-anyon gas displays similar behavior as that of Bose gas with the increasing interaction. In the strong interacting regime the density distribution shows the same behavior as the free fermion [33–37], which is irrespective to the statistical parameter. The special property resulted from the fractional statistics exhibits in the reduced one body density matrices and the momentum distributions [38, 39]. The momentum distribution of anyon differs from fermion's oscillations and boson's single peak structure, which are symmetric about the zero momentum. The momentum distribution of anyons is asymmetric when the statistical parameters deviate from the Bose and Fermi limit [27–31]. This special behavior originates from that the reduced one body density matrix is a complex Hermitian one rather than a real one for Boson and Fermion.

While most studies have focused on the static properties of the 1D anyonic gas, its dynamics remains to be investigated. The present paper shall study the dynamics of anyons confined in optical lattice with weak harmonic trap in the hard core limit. In Ref. [28], we have extended the exact numerical method originally used to

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treat hard-core bosons by Rigol *et al.* [40] to deal with the ground state of hard-core anyons (HCA) in optical lattice. Here, we further extend this method to investigate the dynamics of HCA. By evaluating the exact time-dependent one-particle Green's function, we obtain the reduced one body density matrix (ROBDM) and thus the density profiles and the momentum distribution at arbitrary time. The dynamical properties induced by anyonic statistics shall be displayed in the momentum distribution.

The paper is organized as follows. In Sec. II, we give a brief review of 1D anyonic model and introduce the numerical method. In Sec. III, we present the density profiles, momentum distributions, occupation distribution and the occupations of the lowest natural orbital for different statistics parameters. A brief summary is given in Sec. IV.

II. FORMULATION OF THE MODEL AND METHOD

We consider N hard core anyons confined in an optical lattice of L sites with a weak harmonic trap and the second quantized Hamiltonian can be formulated as

$$H = -t\sum_{l=1}^{L}\left(a_{l+1}^{\dagger}a_l + H.C.\right) + \sum_{l=1}^{L}V_l a_l^{\dagger}a_l. \quad (1)$$

Here the harmonic potential $V_l = V_0(l-(L+1)/2)^2$ with the strength of the harmonic trap V_0. The anyonic operator $a_l^{\dagger}$ (a_l) create (annihilate) an anyon on site l and satisfies the generalized commutation relations

$$\begin{aligned} a_j a_l^{\dagger} &= \delta_{jl} - e^{-i\chi\pi\epsilon(j-l)}a_l^{\dagger}a_j, \\ a_j a_l &= -e^{i\chi\pi\epsilon(j-l)}a_l a_j \end{aligned} \quad (2)$$

for $j \neq l$, where the sign function $\epsilon(x)$ gives -1, 0 or 1 depending on whether x is negative, zero, or positive and χ is the parameter related with fractional statistics. The generalized commutation relations reduce to fermionic commutation for $\chi = 0$ and reduce to Bose commutation for $\chi = 1$, while for anyons satisfying fractional statistics χ changes in between them. The hard core interactions between anyons restricts the additional condition $a_l^2 = a_l^{\dagger 2} = 0$ and $\left\{a_l, a_l^{\dagger}\right\} = 1$. In the Hamiltonian t denotes the hopping between the nearest neighbour sites, which can be tuned by changing the strength of optical lattice.

In order to solve the model of hard core anyons, we extend the numerical method to investigate the hard core bosons in optical lattice developed by Rigol *et. al.*. We can map the above model into the polarized fermionic Hamiltonian using the generalized Jordan-Wigner transformation

$$a_j = \exp\left(i\chi\pi\sum_{1\leq s<j} f_s^{\dagger}f_s\right) f_j, \quad (3)$$

$$a_j^{\dagger} = f_j^{\dagger}\exp\left(-i\chi\pi\sum_{1\leq s<j} f_s^{\dagger}f_s\right), \quad (4)$$

where $f_j^{\dagger}$ (f_j) is creation (annihilation) operator for fermions. The above Hamiltonian of system with N anyons is transformed into a fermionic one with $N_F = N$ fermions

$$H_F = -t\sum_{l=1}^{L}\left(f_{l+1}^{\dagger}f_l + H.C.\right) + \sum_{l=1}^{L}V_l f_l^{\dagger}f_l, \quad (5)$$

where the Fermionic operator satisfy the Fermi anticommutation relation

$$\left\{f_i, f_j^{\dagger}\right\} = \delta_{ij}, \{f_i, f_j\} = \left\{f_i^{\dagger}, f_j^{\dagger}\right\} = 0. \quad (6)$$

The original question about anyons now can be investigated by solving the model on the polarized fermions in optical lattice. We can obtain the exact many body wavefunction of polarized fermions with the diagonalized method and therefore the ground state and interesting physical phenomena of hard core anyons.

The equal-time Green's function for the hard core anyons at time τ should be expressed as

$$\begin{aligned} G_{jl}(\tau) &= \left\langle \Psi_{HCA}(\tau)\left| a_j a_l^{\dagger}\right| \Psi_{HCA}(\tau)\right\rangle \\ &= \langle \Psi_F(\tau)|\exp\left(i\chi\pi\sum_{\beta}^{j-1} f_{\beta}^{\dagger}f_{\beta}\right) f_j f_l^{\dagger} \\ &\quad \times \exp\left(-i\pi\sum_{\gamma}^{l-1} f_{\gamma}^{\dagger}f_{\gamma}\right)|\Psi_F(\tau)\rangle \\ &= \left\langle \Psi_F^A(\tau)|\Psi_F^B(\tau)\right\rangle \end{aligned} \quad (7)$$

with

$$\begin{aligned} \left\langle \Psi_F^A(\tau)\right| &= \left(f_j^{\dagger}\exp\left(-i\chi\pi\sum_{\beta}^{j-1} f_{\beta}^{\dagger}f_{\beta}\right)|\Psi_F(\tau)\rangle\right)^{\dagger}, \\ \left|\Psi_F^B(\tau)\right\rangle &= f_l^{\dagger}\exp\left(-i\chi\pi\sum_{\gamma}^{l-1} f_{\gamma}^{\dagger}f_{\gamma}\right)|\Psi_F(\tau)\rangle. \end{aligned}$$

Here $|\Psi_{HCA}(\tau)\rangle$ is the wavefunction at time τ of hard core anyons in a system with Hamiltonian H_{HCA}, and $|\Psi_F(\tau)\rangle$ is the corresponding one for the equivalent polarized fermions. For the polarized fermions the time evolution of their initial wavefunction $\left|\Psi_F^I\right\rangle$

$$|\Psi_F(\tau)\rangle = e^{-i\tau H_F/\hbar}\left|\Psi_F^I\right\rangle. \quad (8)$$

While the matrix representation of initial wavefunction can be expressed as

$$\left|\Psi_F^I\right\rangle = \prod_{n=1}^{N_f}\sum_{l=1}^{L} P_{ln}^I f_l^\dagger \left|0\right\rangle \tag{9}$$

so that

$$\begin{aligned}\left|\Psi_F(\tau)\right\rangle &= e^{-i\tau H_F/\hbar}\prod_{n=1}^{N_f}\sum_{l=1}^{L} P_{ln}^I f_l^\dagger \left|0\right\rangle \\ &= \prod_{n=1}^{N_f}\sum_{l=1}^{L} P_{ln}(\tau) f_l^\dagger \left|0\right\rangle .\end{aligned}$$

Thus the fermonic time-dependent wavefunction can be expressed as an $L \times N_f$ matrix $\mathbf{P}(\tau)$. The matrix $\mathbf{P}(\tau)$ can be evaluated as

$$e^{-i\tau H_F/\hbar}\mathbf{P}^I = Ue^{-i\tau D/\hbar}U^\dagger \mathbf{P}^I, \tag{10}$$

where U is an unitary transformation diagonalizing the Hamiltonian H_F, *i.e.*, $U^\dagger H_F U = D$ with diagonal matrix D. After an easy evaluation the state $\left|\Psi_F^A\right\rangle$ reads

$$\left|\Psi_F^A(\tau)\right\rangle = \prod_{n=1}^{N_f+1}\sum_{l=1}^{L} P_{ln}^{\prime A}(\tau) f_l^\dagger \left|0\right\rangle$$

with

$$\begin{aligned}P_{ln}^{\prime A}(\tau) &= \exp(-i\chi\pi) P_{ln}(\tau) \qquad \text{for } l \le j-1 \\ P_{ln}^{\prime A}(\tau) &= P_{ln}(\tau) \qquad\qquad\qquad \text{for } l \ge j\end{aligned}$$

for $n \le N_f$ and $P_{jN_f+1}^{\prime A}(\tau) = 1$ and $P_{lN_f+1}^{\prime A}(\tau) = 0$ $(l \neq j)$. The state $\left|\Psi_F^B\right\rangle$ has the same form with the replace of j by l. The time-dependent Green's function is a determinant dependent on the $L \times (N_f+1)$ matrixes $\mathbf{P}^{\prime A}(\tau)$ and $\mathbf{P}^{\prime B}(\tau)$

$$G_{jl}(\tau) = \left\langle \Psi_F^A(\tau) | \Psi_F^B(\tau)\right\rangle = \det\left[\left(\mathbf{P}^{\prime A}(\tau)\right)^T \mathbf{P}^{\prime B}(\tau)\right].$$

In the present paper we will focus on the time evolution of the density profile and momentum distribution for hard core anyons with different statistical parameter χ. The ROBDM can be evaluated by Green's function

$$\rho_{jl}(\tau) = \left\langle a_j^\dagger a_l \right\rangle = \delta_{jl}(1 - G_{jl}(\tau)) - (1-\delta_{jl})e^{-i\chi\pi}G_{jl}(\tau).$$

The diagonal part of ROBDM is the density profile and its Fourier transformation is defined as momentum distribution

$$n(k) = \frac{1}{2\pi}\sum_{j,l=1}^{L} e^{-ik(j-l)}\rho_{jl}(\tau). \tag{11}$$

The natural orbitals ϕ^η are defined as the eigenfunctions of the one-particle density matrix

$$\sum_{j=1}^{L}\rho_{jl}\phi^\eta = \lambda_\eta \phi^\eta, j = 1, 2, ...L, \tag{12}$$

and can be understood as the effective single-particle states with occupations λ_η.

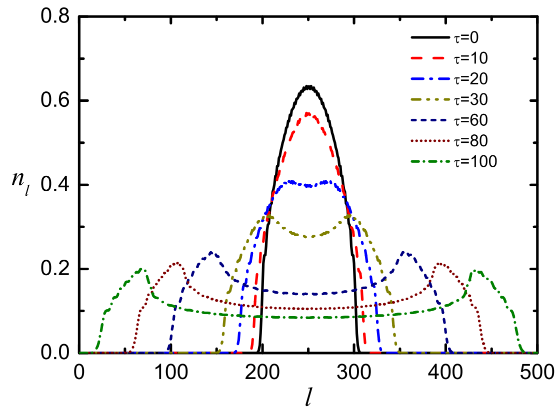

FIG. 1: (color online) The evolving of density distribution for 50 hard core anyons in optical lattice of 500 sites. $V_0^I = 1.0 \times 10^{-3}t$ and $V_0 = 1.0 \times 10^{-8}t$.

III. DYNAMICS OF DENSITY PROFILE AND MOMENTUM DISTRIBUTION

In the present paper we investigate the dynamics of hard core anyons in optical lattice. Initially we confine the anyons in optical lattice with a strong harmonic trap and then turn off the harmonic trap or reduce the strength of harmonic trap. The trapped anyons will evolve in the optical lattice. For convenience we set lattice constant a as 1, the unit of k is $1/a$ and the unit of time is $\hbar/t$.

In Fig. 1, 2, 3 and 4 we show the evolution of 50 anyons in optical lattice of 500 sites. Initially the strength of harmonic trap $V_0^I = 1.0 \times 10^{-3}t$ and at $\tau = 0$ the harmonic trap is released to $V_0(0) = 1.0 \times 10^{-8}t$, i.e., the anyons are trapped only in optical lattice. The evolution of density distribution is shown in Fig. 1. It turns out that the density distribution of anyons shall not display any different dynamical property from those of bosons and fermions. We cannot distinguish the statistical properties by the density distribution in real space. Similar to hard core bosons and polarized fermions, anyons expand in the optical lattice and they shall gradually populate in the full lattice. The density of anyons at the center reduces faster than the density at the border because the anyons locating in the sites of high density posses higher energy. As evolving time is long enough anyons homogeneously distribute in the middle regime while its density distribution shows peaks at the border regime.

The corresponding momentum distribution is displayed in Fig. 2. Initially bosons and fermions distribute symmetrically about the zero momentum as respective statistical property and anyons $(0 < \chi < 1)$ exhibit the asymmetrical momentum distribution. After the harmonic trap is turned off for anyons of different statistical parameter their momentum distribution show different evolving properties. Fermions $(\chi = 0.0)$ do not show

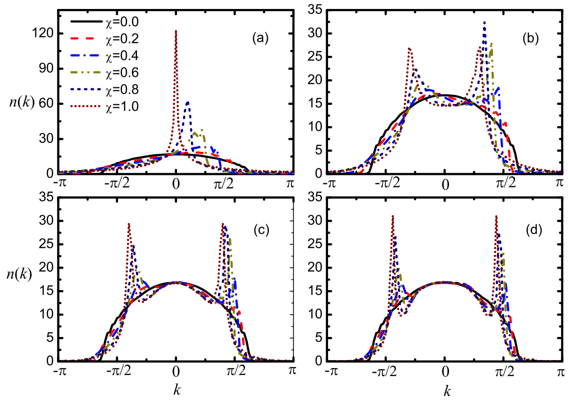

FIG. 2: (color online) The momentum distributions for 50 hard core anyons in lattice of 500 sites, $V_0^I = 1.0 \times 10^{-3}t$ and $V_0 = 1.0 \times 10^{-8}t$. (a) $\tau = 0$, (b) $\tau = 20$, (c) $\tau = 50$,(d) $\tau = 100$.

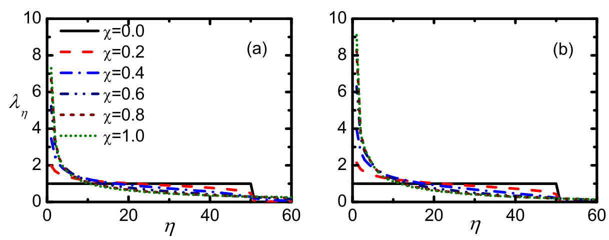

FIG. 3: (color online) The occupation distributions for 50 hard core anyons in lattice of 500 sites, $V_0^I = 1.0 \times 10^{-3}t$ and $V_0 = 1.0 \times 10^{-8}t$. (a) $\tau = 0$, (b) $\tau = 100$.

obvious change of momentum distribution (In fact the slight change happens according to the numerical data). As the statistical parameter deviates from $\chi = 0$ the momentum distribution of anyon evolves from the asymmetrical structure of single peak to the structure similar to that of Fermions but with two asymmetrical peaks. As statistical parameter increases these two peaks become more and more obvious. For the case of $\chi = 1$ (hard core bosons) the momentum distribution evolve to the structure of symmetrical double peaks. When the evolving time is long enough the momentum distribution of anyons of any statistical parameter ($0 \leq \chi \leq 1$) exhibits identical behavior in the middle regime.

In Fig. 3 we show the occupation distribution of natural orbitals for the same system as above. In the full evolution period the occupation distribution of each orbital does not change qualitatively. In the Bose limit anyons occupy the lower natural orbitals and the occupation distribution displays the single peak structure. In the Fermi limit each anyon occupies one natural orbital and at any time the occupation distribution seems like a

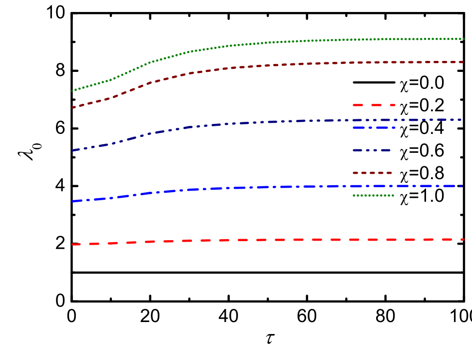

FIG. 4: (color online) The occupations of the lowest natural orbital for 50 bosons in lattice of 500 sites. $V_0^I = 1.0 \times 10^{-3}t$ and $V_0 = 1.0 \times 10^{-8}t$.

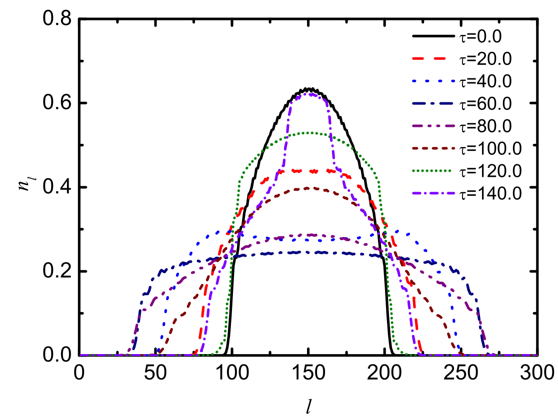

FIG. 5: (color online) The density distribution of 50 hard core anyons in optical lattice of 300 sites. $V_0^I = 1.0 \times 10^{-3}t$ and $V_0 = 2.0 \times 10^{-4}t$.

step-function. While for anyons in between these two limits, the occupations of higher natural orbitals increase as the statistical parameter evolves from Bose limit to Fermi limit. We also display the evolution of the occupation of the lowest natural orbital in Fig. 4. It is shown that the occupation is time independent in the Fermi limit and as deviating from the Fermi limit the occupation increases during the time evolution. The increase is more obvious for the bigger statistical parameter. When the evolving time is long enough the occupation of the lowest natural orbital shall preserve a constant.

If the initially confined hard core anyons in optical lattice evolve in a weak harmonic trap rather than turning off the harmonic potential, the situation will be different. Anyons with different statistical parameters always exhibit the same density distributions and we also cannot determine the statistical properties according to evolving property of density profiles. In Fig. 5 we display the evolving density distribution of 50 anyons in optical

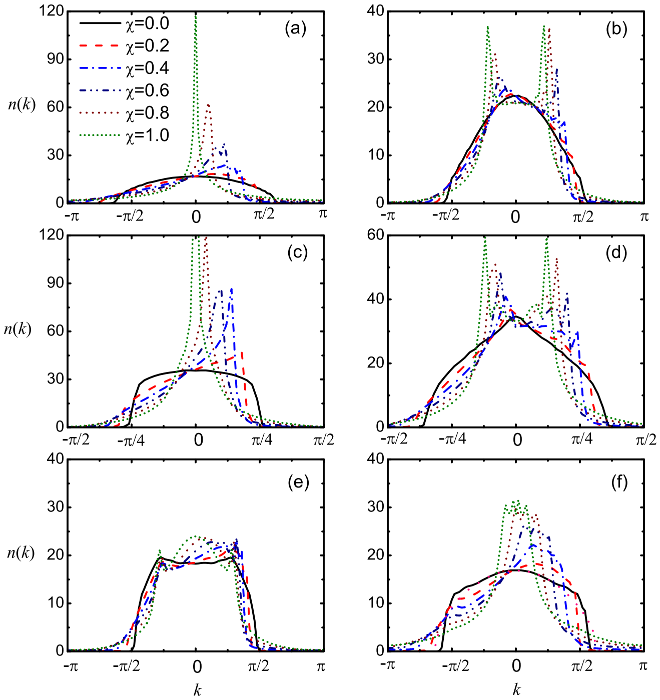

FIG. 6: (color online) The momentum distribution of 50 anyons in optical lattice of 300 sites. $V_0^I = 1.0 \times 10^{-3}t$ and $V_0 = 2.0 \times 10^{-4}t$. (a) $\tau = 0$, (b) $\tau = 30$, (c) $\tau = 60$, (d) $\tau = 80$, (e) $\tau = 100$, (f) $\tau = 120$.

lattice of 300 sites combined with a weaker harmonic potential ($V_0 = 2.0 \times 10^{-4}t$). Initially, anyons distribute in the middle regime of the harmonic trap and after the harmonic potential becomes weak anyons shall expand firstly. The central density decreases faster than the boundary density such that the density profiles behave as a Fermi-like distribution at $\tau = 60$. Then anyons shall contract because of the confinement of harmonic trap. With the time evolution anyons redistribute in the middle regime of the trap.

During the expansion, momentum distributions show rich dynamical structure (Fig. 6). In the Bose limit, the momentum distribution firstly evolves from the original structure of a single peak to the structure of double peaks, and then back to the structure of a single peak at $\tau = 60$. During the later period momentum distribution shall behave as the single peak and double peaks alternately. In the Fermi limit the momentum distribution does not keep its original profile as shown in Fig. 2 and shall exhibit the oscillating behavior. Contrary to the evolution of density distribution, the momentum distribution contracts firstly and at $\tau = 60$ (Fig. 6c) displays the step-function profile in the region of $-\pi/4 \leq k \leq \pi/4$. Then it gradually expands to the region of higher momentum and at $\tau = 120$ (Fig. 6f) momentum distribution almost recovers to the behavior at $\tau = 0.0$. For anyon

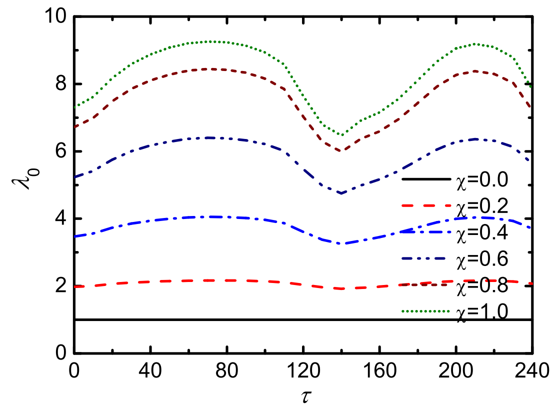

FIG. 7: (color online) The occupation of the lowest natural orbital for 50 anyons in optical lattice of 300 sites. $V_0^I = 1.0 \times 10^{-3}t$ and $V_0 = 2.0 \times 10^{-4}t$.

gas in between ($0 < \chi < 1$) its asymmetric momentum distribution also shows the alternate behavior. For the case of bigger statistical parameter asymmetric double peaks are displayed and for the case of smaller statistical parameter momentum distribution exhibits the behavior similar to those of fermions.

The evolution of occupation for the anyons in optical lattice with weak harmonic potential is similar to the case only confined in optical lattice. The occupation distribution always exhibits the same behaviors qualitatively as those at initial time. But the occupation of the lowest natural evolve with time, which is displayed in Fig. 7. It is shown that in the Fermi limit the occupation is time-independent and as deviating from the Fermi limit it shall oscillate with the time evolution. It does not increase monotonously rather than fluctuates with the time evolution. The bigger the statistical parameter, the stronger the oscillation amplitude.

In order to investigate the dynamical properties of anyons in Mott regime, we initially prepare a Mott state by superimposing optical lattice with a tight harmonic potential and then turn off it at $\tau = 0.0$. The dynamical evolutions of Mott state are displayed in Fig. 8. Initially, the momentum distribution for anyons of any statistical parameter exhibits the behavior similar to that of fermions. There are only tiny differences in the regime close to zero momentum. After the harmonic trap is turned off, the momentum distribution of anyons in the Fermi limit always preserve its initial profile. Anyons deviating this limit still display Fermi-like momentum distribution in the full momentum regime, but the structure of double peaks appears at particular regime. In the full evolving period anyons in Bose limit show the sharpest peaks and as statistical parameter decreases (close to the Fermi limit) the peaks become smaller. It is same as before that anyons always exhibit the asymmetrical momentum distributions except of the Bose limit and Fermi

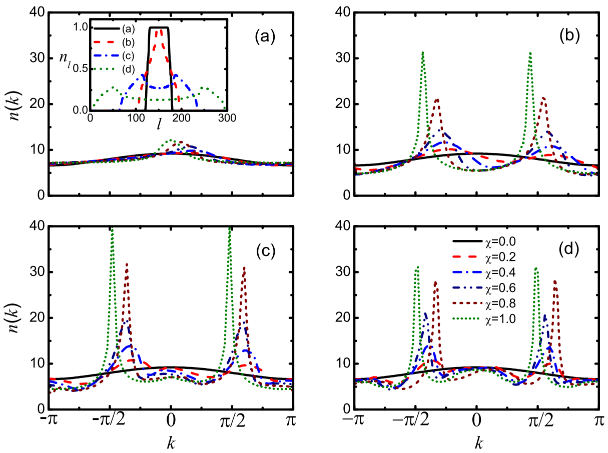

FIG. 8: (color online) The momentum distribution of 50 anyons in lattice of 300 sites. $V_0^I = 1.0 \times 10^{-2}t$ and $V_0 = 1.0 \times 10^{-8}t$. (a) $\tau = 0$, (b) $\tau = 10$, (c) $\tau = 30$,(d) $\tau = 60$. Inset: Density distributions for the same system.

limit. We also display the evolving density profiles in the inset, which is independent of the statistical parameter. At the beginning anyons are confined in the central regime with an anyon per site. After the harmonic trap is turned off, anyons expand in the lattice and peaks appear at the boundary regime.

IV. CONCLUSIONS

In summary, in the present paper we have developed the exact numerical method to deal with the dynamics of hard-core anyons confined in the optical lattice superimposed with a weak harmonic potential. By evaluating the exact time-dependent one-particle Green's function, we obtain the density profiles, momentum distributions, occupation distribution, and the occupation of the lowest natural orbital in the full evolving period. It is shown that the evolving property of density profiles is independent on the statistical parameter of anyons. As the harmonic trap is turned off anyon shall expand in the optical lattice and distribute gradually in the full lattice. The dynamical properties of momentum distributions and occupation distributions of natural orbitals depend on statistical parameter of anyons. In the Bose limit, momentum distribution always displays the structure of double peaks. As deviating from the Bose limit, the double peaks become smaller and in the Fermi limit the double peaks disappear. When the evolving time is long enough, anyons display the same momentum distribution irrespective of statistical parameter in the regime nearby zero momentum. If the anyon gas is initially in the Mott regime, during the full evolving period it shall be always in the Mott regime but at the particular momentum position double peaks appear. When the harmonic trap is replaced with a weaker one, the density profiles of anyons shall exhibit the breathing behavior. The momentum distributions of bosons alternatively display the structure of single peak and the structure of double peaks, while fermions contract and expand in the k-space with the time evolution. For anyon interpolating between these two limit, momentum distribution also repeats its behavior at regular intervals dependent on the statistical parameter. The interval time is the longest for fermions and is the shortest for bosons. The occupation distributions of natural orbitals do not change qualitatively with the time evolution but the occupation of the lowest natural orbital evolves with time for anyon gas deviating the Fermi limit. When the harmonic trap is turned off it increases and gradually arrives at the biggest value. When the harmonic trap becomes weaker rather than being turned off, it shall fluctuate with the time evolution.

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